Numerical Solutions of Two-Dimensional Burgers' Equations

Vildan Gülkaç-Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli/Turkey vgulkac@kocaeli.edu.tr

Abstract— Two-dimensional Burgers' equations are reported various kinds of phenomena such as turbulence and viscous fluid. In this paper, we illustrate the LOD method for solving the two-dimensional coupled Burgers' equations. We extend our earlier work [1] and a stability analysis by Fourier method of the LOD method is also investigated. The computational results obtained by present method are in excellent agreement with earlier results. Present method can be easily implemented for solving nonlinear problems evolving in several branches of engineering and science.

Keywords— Burgers' equations, Finite-difference, LOD method, Reynolds number

1. Introduction

Burgers' equation is a fundamental partial differential equation from fluid mechanics. It has been found to describe various kinds of phenomena such as modeling of dynamics, heat conduction, acoustic waves, a mathematical model of turbulence, and the approximate theory of flow through a shock wave traveling in a viscous fluid.

Consider two-dimensional coupled nonlinear Burgers' equations taken from [2]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

subject to the conditions

$$u(x,y,0)=f(x,y), x,y\in D$$
$$v(x,y,0)=g(x,y), x,y\in D \quad (3)$$

and boundary conditions

$$u(x,y,t)=f_1(x,y,t), x,y\in\partial D, t>0$$
$$v(x,y,t)=g_1(x,y,t), x,y\in\partial D, t>0 \quad (4)$$

Here $D=\{(x,y): a\le x\le b, a\le y\le b\}$, ∂D denotes the boundary of D, u(x,y,t) and v(x,y,t) are the velocity components to be determinant f,g and f_1, f_2 are known functions and R is Reynolds number.

The analytic solution of equations (1) and (2) were proposed by Fletcher using the Hope-Cole transformation [3]. The numerical solutions of this equation system have been studied by several authors. Jain and Holla [4] developed two algorithms based on cubic spline technique. Fletcher [5] has discussed the comparison of a number of different numerical approaches. Goyon [6] used several multilevel schemes with ADI. A.R Bahadır [2] has applied a fully implicit method. V.K. Srivastava et al. [7] has applied a Crank-Nicolson scheme, El-Sayed and Kaya has applied a decomposition method [8], Zhu et all. [9] developed numerical solutions by discreate Adomian decomposition method.

In this paper, Locally One Dimensional (LOD) method is used to solve two-dimensional Burgers' equations. Computed results are compared with analytical and other numerical results.

2. LOD Method and Adaptation of Solution Methodology

Adaptation of solution methodology numerical computations is always active areas for solutions of differential equations. The finite-difference methods are easy to use for numerical solutions, this methods are still used extensively in practical computations. Recently, there have been a renewed interests in the worked and the application of finite-difference methods for the solutions of the multi-dimensional partial differential equations [2, 7].

Today, new difference methods have been constantly presented and for multi dimensional problems LOD and ADI scheme get much attention for their unconditional stability and high efficiency. Gülkaç [1, 10] suggested a LOD method for the solution of multi dimensional phase change problems.

The two dimensional Burgers' equations (1) and (2) can be written by splitting it into two one- dimensional equations, respectively eqns. (5), (6) and (7), (8), as seen Gülkaç [1].

The domain of definition was separated into sets of sub domains defined along the x and y variable mesh such as equations (5) and (6). Each of the equations was then solved over half of the time step used for the complete two-dimensional equation using techniques for the one-dimensional problems.

$$\frac{1}{2}\frac{\partial u}{\partial t}=\frac{1}{2R}\left\{\left(\frac{\partial^2 u}{\partial x^2}\right)_{i+1,j}+\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\right\}-\frac{1}{2}u\left(\frac{\partial u}{\partial x}\right)_{i,j}-\frac{1}{2}v\left(\frac{\partial u}{\partial y}\right)_{i,j} \quad (5)$$

and

$$\frac{1}{2}\frac{\partial u}{\partial t}=\frac{1}{2R}\left\{\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j+1}+\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}\right\}-\frac{1}{2}u\left(\frac{\partial u}{\partial x}\right)_{i,j}-\frac{1}{2}v\left(\frac{\partial u}{\partial y}\right)_{i,j} \quad (6)$$

similarly,

$$\frac{1}{2}\frac{\partial v}{\partial t}=\frac{1}{2R}\left\{\left(\frac{\partial^2 v}{\partial x^2}\right)_{i+1,j}+\left(\frac{\partial^2 v}{\partial x^2}\right)_{i,j}\right\}-\frac{1}{2}u\left(\frac{\partial v}{\partial x}\right)_{i,j}-\frac{1}{2}v\left(\frac{\partial v}{\partial y}\right)_{i,j} \quad (7)$$

and

$$\frac{1}{2}\frac{\partial v}{\partial t}=\frac{1}{2R}\left\{\left(\frac{\partial^2 v}{\partial y^2}\right)_{i+1,j}+\left(\frac{\partial^2 v}{\partial y^2}\right)_{i,j}\right\}-\frac{1}{2}u\left(\frac{\partial v}{\partial x}\right)_{i,j}-\frac{1}{2}v\left(\frac{\partial v}{\partial y}\right)_{i,j} \quad (8)$$

In this method, we replace all spatial derivatives with the average of their values at the n and $n+1/_2$ time levels and then substitute the central finite form all derivatives. Eqns. (5), (6) and (7), (8) can be written as

$$\frac{1}{2}\left(\frac{u_{i,j}^{n+1/2}-u_{i,j}^{n}}{\Delta t}\right)=\frac{1}{2R\Delta x^2}\left\{\delta_x^2-\frac{1}{12}\delta_x^4+\frac{1}{90}\delta_x^6\mp\cdots\right\}\left(u_{i+1,j}^{n}+u_{i,j}^{n}\right)$$

$$-\frac{1}{2}u\left(\frac{\partial u}{\partial x}\right)_{i,j}-\frac{1}{2}v\left(\frac{\partial u}{\partial y}\right)_{i,j}$$

or

$$\left\{1+\left(\frac{1}{12}-\frac{1}{2}r_x\right)\delta_x^2\right\}u_{i,j}^{n+1/2}=\left\{1+\left(\frac{1}{12}+\frac{1}{2}r_x\right)\delta_x^2\right\}u_{i,j}^{n}$$

$$-\frac{1}{2}u_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}-\frac{1}{2}v_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta y} \quad (9)$$

for $\forall i=1,\dots,N,\forall j=1,\dots N, r_x=\Delta t/\Delta x^2, r_y=\Delta t/\Delta y^2$ then $t_{n+1/2}\rightarrow t_{n+1}$ and equation (6) can be written as,

$$\left\{1+\left(\frac{1}{12}-\frac{1}{2}r_y\right)\delta_y^2\right\}u_{i,j}^{n+1}=\left\{1+\left(\frac{1}{12}+\frac{1}{2}r_y\right)\delta_y^2\right\}u_{i,j}^{n+1/2}-\frac{1}{2}u_{i,j}^{n}\frac{u_{i,j}^{n+1/2}-u_{i-1,j}^{n+1/2}}{\Delta x}-$$

$$\frac{1}{2}v_{i,j}^{n}\frac{u_{i,j}^{n+1/2}-u_{i,j-1}^{n+1/2}}{\Delta y} \quad (10)$$

Equations (7) and (8) can be written similarly as equation (9) and (10).

We consider the use of equal mesh spacing $\Delta x=\Delta y$ over each sub domain for the problem. It should be noted that the solution algorithm possesses high flexibility for using unequal mesh spacing provided that the stability of equations are valid for each spatial mesh spacing separately.

3. Stability of Equations

The basic idea defining von Neumann stability [11] is that this numerical algorithm used exactly, should limit the amplification of all elements of initial conditions.

The Burgers' equation express the initial values at the mesh points along t=0 in terms of a finite Fourier series, then regards the growth a function that reduces to this series for t=0 by a variable separable method indistinguishable to that commonly used for solving partial differential equations. The Fourier series can be formulated in complex exponential form [11].

In order to show the von Neumann stability of the present method, we replace

$$u_{i,j,n}=u_{p,q,r}=e^{i\beta ph}e^{i\beta qk}e^{i\alpha t}=e^{i\beta ph}e^{i\beta qk}\xi^{r}$$

and $r_x=R_x, r_y=R_y$.

Where $\xi=e^{\alpha t}, i=\sqrt{-1}$ and α in general is a complex constant, ξ is often called amplification factor [11]. The finite difference equations will be stable by von Neumann definition if $|\xi|\le 1$ [11].

Equation (9) can be written as

$$\left(\frac{1}{12}-\frac{1}{2}R_x\right)e^{i\beta(p+1)h}e^{i\beta qk}\xi^{r+1/2}+(R_x-5)e^{i\beta ph}e^{i\beta qk}\xi^{r+1/2}+\left(\frac{1}{12}-\frac{1}{2}R_x\right)e^{i\beta(p-1)h}e^{i\beta qk}\xi^{r+1/2}=\frac{1}{R}\left\{(-5-R_x)e^{i\beta ph}e^{i\beta qk}\xi^{r}+\left(\frac{1}{12}+\frac{1}{2}R_x\right)e^{i\beta(p+1)h}e^{i\beta qk}\xi^{r}+\left(\frac{1}{12}+\frac{1}{2}R_x\right)e^{i\beta(p-1)h}e^{i\beta qk}\xi^{r}\right\}-\frac{1}{2\Delta x}u_0\left(e^{i\beta ph}e^{i\beta qk}\xi^{r}-e^{i\beta(p-1)h}e^{i\beta qk}\xi^{r}\right)-\frac{1}{2\Delta y}v_0\left(e^{i\beta ph}e^{i\beta qk}\xi^{r}-e^{i\beta ph}e^{i\beta(q-1)k}\xi^{r}\right) \quad (11)$$

u_0 *and* v_0 initial conditions then we let $u_0=v_0=0$ and $\Delta x=\Delta y$, and division eqn.(11) by $e^{i\beta ph}e^{i\beta qk}\xi^{r}$ equation (11) can be written as,

$$\xi^{1/2}\left\{\left(\frac{1}{12}-\frac{1}{2}R_x\right)e^{-i\beta h}+(R_x-5)+\left(\frac{1}{12}-\frac{1}{2}R_x\right)e^{i\beta ph}\right\}=\frac{1}{R}\left\{\left(\frac{1}{12}+\frac{1}{2}R_x\right)e^{-i\beta ph}-(R_x+5)+\left(\frac{1}{12}+\frac{1}{2}R_x\right)e^{i\beta ph}\right\}$$

and then

$$\xi^{1/2}=\frac{\left(\frac{1}{12}+\frac{1}{2}R_x\right)2\cos\beta h-(R_x+5)}{R\left\{\left(\frac{1}{12}+\frac{1}{2}R_x\right)2\cos\beta h\right\}+(R_x-5)} \quad (12)$$

let $\xi^{1/2} = K$ the condition is $|K| \leq 1$.This required. Clearly,

$0 < K \leq 1$ $for\ all\ R_x > 0\ and\ all\ \beta\ and\ R = Reynols\ number \geq 1,\ |K| \leq 1.$

Therefore, equations unconditionally stable.

Similarly, it is easily shown by same method that equations (10) and (7), (8) are unconditionally stable.

4. Numerical Examples and Conclusions

4.1 Problem I

The exact solutions of Burgers' equations (1) and (2)can be generated by using the Hope-Cole transformation [3] which are:

$$u(x,y,t) = \frac{3}{4} - \frac{1}{4[1+\exp((-4x+4y-t)R/32]},$$

$$v(x,y,t) = \frac{3}{4} + \frac{1}{4[1+\exp((-4x+4y-t)R/32]},$$

Here the computational domain is taken as a square domain $D = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$. The initial and boundary conditions are taken from the exact solutions. The numerical computations are performed using uniform grid, with a mesh width $\Delta x = \Delta y = 0.05$.

From Tables 1-4, it is clear that the results from the present study are in good agreement with the exact solution for different values of Reynolds number (R) and some typical mesh points demonstrate that the present scheme achieves similar results as those of Jain and Holla [4], Bahadır [2], Srivastava et all [7].

4.2. Problem II

Here the computational domain is taken as

$$D = \{(x,y): 0 \leq x \leq 0.5, 0 \leq y \leq 0.5\}$$

and Burgers' equation (1) and (2) are taken with the initial conditions,

$$\left.\begin{array}{l} u(x,y,0) = sin\pi x + cos\pi y \\ v(x,y,0) = x + y \end{array}\right\} 0 \leq x \leq 0.5, 0 \leq y \leq 0.5$$

and boundary conditions,

$$\left.\begin{array}{l} u(0,y,t) = cos\pi y, u(0.5,y,t) = 1 + cos\pi y \\ v(0,y,t) = y, \qquad v(0.5,y,t) = 0.5 + y \end{array}\right\} 0 \leq y \leq 0.5, t \geq 0,$$

$$\left.\begin{array}{l} u(x,0,t) = 1 + sin\pi x, u(x,0.5,t) = sin\pi x \\ v(x,0,t) = x, \qquad v = x,0.5,t) = x + 0.5 \end{array}\right\} 0 \leq x \leq 0.5, t \geq 0$$

The numerical methodology used is similar to that of Gülkaç [1] and Gülkaç and Öziş [10]. We presented the numerical method for solving two-dimensional Burgers' equations using the LOD method and then substituted Douglas-like equation form for all derivatives [1, 11].

Equations (1), (2) are discredited using the LOD method. The stability analysis of the scheme is also investigated and the scheme is therefore unconditionally stable. The accuracy of the numerical solutions indicates that the method is well suited for the solution of two-dimensional non-linear Burgers' equations.

Table 1

Comparison of numerical values of u for R=500 at t=0.625

(x, y)	Numerical values of u			
	Present work	Bahadır [2]	Jain and Holla [3]	Srivastava et all. [4]
	N=20	N=20	N=20	N=20
(0.15, 0.1)	0.96657	0.96650	0.95691	0.96870
(0.3, 0.1)	1.02977	1.02970	0.95616	1.03200
(0.1, 0.2)	0.84456	0.84449	0.84257	0.86178
(0.2, 0.2)	0.87638	0.87631	0.86399	0.86178
(0.1, 0.3)	0.67816	0.67809	0.67667	0.67920
(0.3, 0.3)	0.79799	0.79792	0.76876	0.79947
(0.15, 0.4)	0.54609	0.54601	0.54408	0.66036
(0.2, 0.4)	0.58881	0.58874	0.58778	0.58959

Table 2

Comparison of numerical values of v for R=500 at t=0.625

(x, y)	Numerical values of v			
	Present work	Bahadır [2]	Jain and Holla [3]	Srivastava et all. [4]
	N=20	N=20	N=20	N=20
(0.15, 0.1)	0.09027	0.09020	0.10177	0.09043
(0.3, 0.1)	0.10697	0.10690	0.13287	0.10727
(0.1, 0.2)	0.17979	0.17972	0.18503	0.17295
(0.2, 0.2)	0.16784	0.16777	0.18169	0.16816
(0.1, 0.3)	0.26231	0.26222	0.26560	0.26268
(0.3, 0.3)	0.23504	0.23497	0.25142	0.23550
(0.15, 0.4)	0.31761	0.31753	0.32084	0.29019
(0.2, 0.4)	0.30379	0.30371	0.30927	0.30419

Table 3

Comparison of numerical values of u for R=50 at t=0.625

(x, y)	Numerical values of u			
	Present work	Bahadır [2]	Jain and Holla [3]	Srivastava et all. [4]
	N=20	N=20	N=20	N=20
(0.1, 0.1)	0.96695	0.96668	0.97258	0.97146
(0.3, 0.1)	1.14835	1.14827	1.16214	1.15280
(0.2, 0.2)	0.85918	0.85911	0.86281	0.86307
(0.4, 0.2)	0.97644	0.97637	0.96483	0.97981
(0.1, 0.3)	0.66026	0.66019	0.66318	0.66316
(0.3, 0.3)	0.76939	0.76932	0.77030	0.77230
(0.2, 0.4)	0.57974	0.57966	0.58070	0.58180
(0.4, 0.4)	0.75686	0.75678	0.74435	0.75856

Table 4

Comparison of numerical values of v for R=50 at t=0.625

(x, y)	Numerical values of v			
	Present work	Bahadır [2]	Jain and Holla [3]	Srivastava et all. [4]
	N=20	N=20	N=20	N=20
(0.1, 0.1)	0.09832	0.09824	0.09773	0.09869
(0.3, 0.1)	0.14119	0.14112	0.14039	0.14158
(0.2, 0.2)	0.16689	0.16681	0.16660	0.16754
(0.4, 0.2)	0.17073	0.17065	0.17397	0.17110
(0.1, 0.3)	0.26269	0.26261	0.26940	0.26378
(0.3, 0.3)	0.22582	0.22576	0.22463	0.22654
(0.2, 0.4)	0.32754	0.32745	0.32402	0.32851
(0.4, 0.4)	0.32447	0.32441	0.31822	0.32500

5. References

[1] V. Gülkaç, A numerical solution of two-dimensional fusion problem with convective boundary conditions, International Journal for Computational Methods in Engineering Science and Mechanics, 11 (2010) 2-26.

[2] A.R. Bahadır, A fully implicit finite-difference scheme for two-dimensional Burgers' equations, Appl. Math. Comput. 206 (2008) 131-137.

[3] C.A.J. Fletcher, Generating exact solutions of two-dimensional Burgers' equation, Int. J. Numer. Meth. Fluids 3 (1983) 213-216.

[4] P.C. Jain, D.N. Holla, Numerical solutionof coupled Burgers' equations, Int. J. Numer. Meth. Eng. 12 (1978) 213-222.

[5] C.A.J. Fletcher, A comparison of finite element and finite difference solution of the one-and two-dimensional Burgers' equations, J. Comput. Phys. 51 (1983) 159-188.

[6] O. Goyon, Multilevel schemes for solving unsteady equations, Int. J. Numer. Meth. Fluids 22 (1996) 937-959.

[7] M.K. Srivastava, M. Tamsir, U. Bhardwaj, Y. Sanyasiraju, Crank-Nicolson scheme for numerical solutions of two dimensional coupled Burgers' equations, 2 (2011) 1-7.

[8] S.M. El-Sayed, D. Kaya, On the numerical solution of the system of two-dimensional Burgers' equations by decomposition method, 158 (2004) 101-109.

[9] H. Zhu, H. Shu, M. Ding, Numerical solutions of two-dimensional Burgers' equations by discreate Adomian decomposition method, Computer and Mathematics with Applications, 60 (2010) 840-848.

[10]V. Gülkaç, T. Öziş, On a LOD method for solution of two-dimensional Fusion problem with convective boundary conditions, International Communications in Heat and Mass Transfer, 31 (2004) 597-606.

[11] G.D. Smith, Numerical Solutions of Partial Differential Equations: Finite Difference Methods, Clarendon Press, Oxford, 1985.